Kaluza-Klein gravitoelectromagnetism and dark matter

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Abstract

Gravitational alternatives to dark matter require additional fields or assumptions beyond general relativity while continuing to agree with tight solar system constraints. Modified Newtonian Dynamics (MOND), for example, predicts the Tully-Fisher relation for galaxies more accurately than dark matter models while limiting to Newtonian gravity in the solar system. On the other hand, MOND does a poor job predicting larger scale observations such as the Cosmic Microwave Background and Matter Power Spectra. Aether-Scalar-Tensor (AeST) theory is a relativistic generalization of MOND that accounts for these observations without dark matter. In this paper, I derive AeST from Kaluza-Klein theory in one extra dimension as a consequence of higher dimensional gravitoelectromagnetism or "frame dragging". In the KK theory, MOND is a special case of a slicing condition in the 5D ADM formalism. This has two benefits: first is means that AeST is compatible with Kaluza-Klein dark matter theory, which is a strong candidate for Weakly Interacting Massive Particles (WIMPs), the other is that it provides an elegant mechanism for the scalar and vector fields. It constrains most of the freedom in the definition of AeST which does not have a field theoretic motivation. This is important because the Kaluza-Klein theory predicts that spin-2 tensor modes must propagate at the speed of light, in agreement with observation, from theoretical constraints while AeST has to match this observation empirically. Furthermore, it removes need for the interpolating function in MOND and the Lorentz-violating condition on the vector field to be physical since they are analogous to a gauge condition and depend on state of motion.

I. INTRODUCTION

Fritz Zwicky first proposed dark matter to explain the rotational curves of stars in the outer reaches of galaxies which appear to rotate far faster than Newtonian physics predicts [1]. The best evidence for Dark Matter, however, is at the largest scales. The ΛCDM model does an excellent job fitting observations at the cosmological scale as well as weak and strong lensing of galaxy clusters. At smaller scales, however, it runs into difficulty. Simulations and observations of individual galaxies (those with high mass and dwarf satellites in the Local Group) have led to the search for ways to either supplement or replace CDM at those scales [2].

In 1983, Milgrom proposed an alternative explanation for galactic rotation curves, modifying the Newtonian force at very low accelerations so that rotational curves match the Tully-Fisher

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relation [3] which is an empirical fitting to these curves [4][5]. The following year, Milgrom and Bekenstein published a modified Newtonian theory that became known as Modified Newtonian Dynamics or MOND [6]. Forty years later, MOND continues to be a serious contender to explain at least some dark matter attributed observations. A recent review of dark matter theories including MOND can be found in [7]. MOND has been shown to fit not only galactic rotation curves [8], better than halo models but also dwarf galaxies such as a recent study in the Fornax cluster [9].

MOND is a modification of the force a body experiences under gravity from the standard $F = ma$ to $F = m\mu(a/a_0)a$ such that $\mu(x) \to 1$ for $x \gg 1$, leading to Newtonian dynamics for the solar system as well as clusters of stars that are small enough where gravitational acceleration is significant, and $\mu(x) \to x$ for $x \ll 1$ which applies to galaxies. The parameter a_0 determines the acceleration at which MOND takes over from Newton.

Colliding galaxy clusters such as the Bullet Cluster (1E 0657-558) and galaxy formation in the early universe are prime test cases for both models of CDM and MOND. Early criticism of MOND that it was unable to reproduce asymmetric weak lensing and that, therefore, such lensing was direct evidence of dark matter were based on the spherically symmetric version [10]. Non-spherically symmetric MOND which do not neglect the additive curl field can reproduce nonlocal weak lensing of galaxy clusters to some extent [11][12]. The curl field, however, requires a great deal of fine tuning which is difficult to justify and may have other unintended effects. Another problem is that when accelerations are close to or higher than a_0, for example Abell 1689 [13], behavior is highly sensitive to the interpolating function μ or MOND predicts behavior should be Newtonian while lensing shows it is not. The case for MOND at these scales is bad.

MOND is not perfect at the galactic scale either. Discrepancies between rotation curves and lensing must be accounted for by introducing gravitational effects beyond those that come from the visible baryonic matter using standard, spherical boundary conditions [14][15]. MOND also overpredicts vertical accelerations, perpendicular to the galactic plane, in conflict with Milky Way data [16]. Other dark matter theories that try to address galactic rotation curves with additional fields beyond pure CDM such as superfluid dark matter [2] also suffer from this same problem [17]. As with the Bullet cluster, such studies, however, neglect the background curl field, so their results do not rule those theories out.

The need for additional matter to explain the behavior of several galaxies is not necessarily disputed [18]. MONDian theories, however, can account for a great deal of the phenomena attributed to such particles at the galactic scale without the need for dark matter halo formation [19]. Halo density profiles are fitted to rotation curves for each galaxy but struggle to explain the lack of variation in rotational curves, especially at high mass scales. Simulations show a much larger scatter around the Tully-Fisher relation than observed [20]. For this reason, one would prefer not to give up on MOND completely at the galactic scale but clearly additional features are needed to account for large scale phenomena.

Because MOND does not directly model lensing, it needs a relativistic generalization that is compatible with tight constraints on GR in the Solar System. Aether-Scalar-Tensor (AeST) theory is a generalization of a relativistic realization of MOND, Tensor-Vector-Scalar (TeVeS), [21] [22]. AeST succeeds in two important observations that Bekenstein's original TeVeS theory did not agree with: the propagation speeds of spin-2 tensor modes [23][24] and Baryonic Acoustic Oscillation peaks in the early universe [25] [22]. Indeed, Skordis et al. showed good agreement with all Cosmic Microwave Background (CMB) power spectra peaks for certain choices for the interpolation function, μ [26]. Dustlike evolution of the scalar field in the early universe, which decouples from the vector field, has a similar effect on BAO as dark matter. AeST remains a strong contender for an alternative to dark matter, partly because it introduces sufficient degrees of freedom to mitigate the need for it in some cases. It also shares features with Einstein-Aether theory that may address galaxies clusters including the bullet cluster [27]. AeST provides a potential explanation, also, for recent JWST observations of large, bright galaxies in the very early universe, where vector field perturbations may be a better predictor of their formation than non-self interacting Cold Dark Matter (CDM) models [28][29][30][31][32].

In AeST, as in TeVeS, the Bekenstein-Sanders metric universally couples to matter,

$$g^{ab} = e^{2\phi}\tilde{g}^{ab} + 2\sinh(2\phi)\beta^a\beta^b, \tag{1}$$

where $\tilde{g}^{ab}$ is the Einstein metric, ϕ is the scalar field, and β^a is the vector field (sometimes also given the symbol A^a or $\mathfrak{U}^a$). Each of these has its own action as described in section III as well as constraints on ϕ given by μ and a constraint on β^a such that $\beta^a\beta_a = -1$. Only certain constraints, in a quasi-static approximation, give the correct MOND and Newtonian limits.

An open problem for AeST is that it is an *ad hoc* empirical theory, constructed to match observations. Its scalar field is constrained to obey MONDian physics. Likewise, its vector field is constrained in such a way as to violate Lorentz covariance. This constraint is relatively mild and appears to be related to the arrow of time, but it is unclear why this particular spin-1 field would obey such a constraint while others do not.

Field theoretic explanations have turned up serious problems with the fine tuning of MONDian models, with TeVeS being one of the less problematic ones but still frustratingly opposed to the clear geometric meaning of pure GR [33]. This creates a considerable problem for matching to observations since the actual form of the theory is not known from first principles or other observations but rather fit to the data. For example, the form of μ is undetermined and different interpolating functions between the Newtonian and MOND regimes generate different results, particularly when a large percentage of matter experiences accelerations on the order of a_0. While there are principles explanations for MOND, notably Verlinde's Entropic Gravity (EG) [34][35], these are non-relativistic, and there are none for AeST.

In this paper, I propose a mechanism by which AeST arises from Kaluza-Klein theory. In one extra dimension, additional fields take the form of a scalar lapse function and a vector shift function. Using Arnowitt-Deser-Misner (ADM) [36] formalism, I show that, not only does Kaluza-Klein theory agree with AeST, it predicts that spin-2 tensor modes propagate at the speed of light as demonstrated in observations of GW170817 [37], a necessary but otherwise *ad hoc* modification that had to be made to the original TeVeS in order to agree with observation [22].

Define a $4 + d$ dimensional metric, γ_{AB} where capital Latin letters will refer to indexes $A, B = 0, 1, 2, 3, \ldots, 4 + d - 1$, lower case Latin letters will refer to $4 + d$ dimensional indexes $a, b, c = 0, 1, 2, 3, \ldots, 4 + d - 2$ where x_{4+d-1} is the ADM flow dimension rather than x_0 as in the standard formalism. From this theory arise a vector and scalar field which under certain conformal transformations, couples to matter in nearly the same way as AeST which becomes identical in both the quasi-static limit which leads to MOND as well as perturbations against an FLRW background for spin-2 tensor modes.

II. KALUZA-KLEIN SLICING

Kaluza-Klein theory has historically been used to unify forces with gravity via a disformal relationship, in which case the other forces such as electromagnetism are the result of a choice of slicing in the 4+d-D manifold for d additional dimensions. KK theory has also been used to generate potential dark matter candidates, e.g., bosonic, based on light particles in a compactified dimension[38][39]. The approach taken in this paper proposes that additional fields in Kaluza-Klein theory could be responsible for MOND behavior in galaxies but does not rule out LKP.

For the remainder of the paper, we will assume $d = 1$ and the additional dimension is x_4 but that does not preclude generalizations to more dimensions. The KK metric obeys the following:

$$dS^2 = -\epsilon\alpha^2 d\tau^2 + g_{ab}(dx^a + \beta^a d\tau)(dx^b + \beta^b d\tau) \tag{2}$$

[40] where $\epsilon = 1$ for timelike and $\epsilon = -1$ for spacelike additional dimension $\tau = x_4$.

In order to be well-defined, the ADM formalism for this KK theory requires a slicing condition. This condition describes how the 5D manifold slices into 4D submanifolds and results in coupled equations for the submanifold metric, g_{ab}, shift vector, β^a, lapse function, α.

In our KK-ADM formalism, the μ function in MOND emerges from the slicing condition. In the ADM slicing conditions are chosen based on practical requirements such as avoiding singularities [41] and removing interdependencies in degrees of freedom [42]. Slicing conditions in 3+1-D GR define the set of spacelike hypersurfaces, $\Sigma_3(t)$, with spatial metric γ_{ij} from one moment in time to the next, where time, t, is a global coordinate parameter. We refer to this condition as spacetime synchronization. The slicing condition enforces what is a standard interval of time at each point in a spatial manifold. There is freedom to scale the infinitesimal tick interval on the extra dimension, and this freedom has the nature of a non-dynamical scalar field which we identify with μ.

The rest of the condition is fixed by a constraint on the shift vector which defines the degree of coordinate shift from one thin slice to the next with zero shift vector being a normal coordinate system.

In a $4+1$ KK theory, the same is true but rather a set of submanifolds $\Sigma_4(\tau)$ which include time as a dimension are defined with metric g_{ab} based on a parameter τ which is analogous to time but may be spacelike.

III. DERIVATION OF AETHER-SCALAR-TENSOR THEORY FROM KALUZA-KLEIN

This AeST theory [30] gives the action,

$$\begin{aligned} S_{\text{TeVeS}} = \quad & \frac{1}{16\pi G}\int d^4x\,\sqrt{\hat{g}}\Big[R - \hat{K} + \lambda_\beta(\beta^a\beta_a + 1) - \\ & \mu g^{ab}\partial_a\phi\partial_b\phi - V(\mu)\Big] + S_M[\tilde{g}]. \end{aligned} \tag{3}$$

where

$$\hat{K} = \hat{K}^{cdab}\nabla_c\beta_d\nabla_a\beta_b \tag{4}$$

and

$$\hat{K}^{cdab} = c_1 g^{ca}g^{db} + c_2 g^{cd}g^{ab} + c_3 g^{cb}g^{da} + c_4 g^{db}\beta^c\beta^a. \tag{5}$$

The original TeVeS theory of Bekenstein [21] assumed that c_i were constant, which caused spin-2 tensor modes of GW to not match the propagation of electromagnetic modes [22]. Recent observations of GW170817 by LIGO require the propagation speed of the gravitational tensor c_T to satisfy constraint of $|c_T - 1| < 10^{-15}$ in units where the speed of light in vacuum $c = 1$ [24]. Skordis et al. show that in AeST theory, if $c_{13} = c_1 + c_3$, then,

$$c_T = e^{-4\phi}/(c_{13} - 1) \tag{6}$$

If $c_T = 1$, this implies that,

$$c_{13} = 1 - e^{-4\phi}. \tag{7}$$

We will now arrive at this theory starting from a $D = 4 + 1$ Kaluza-Klein formalism.

Let the non-conformally scaled, Jordan metric have the form:

$$\gamma'_{44} = 1/\alpha^2 + g'_{ab}\beta^a\beta^b, \tag{8}$$

$$\gamma'_{4a} = \beta_a, \quad \gamma'_{ab} = g'_{ab}, \tag{9}$$

and the inverse is:

$$\gamma'^{44} = \alpha^2, \tag{10}$$

$$\gamma'^{4a} = \alpha^2\beta^a, \quad \gamma'^{ab} = g'^{ab} + \alpha^2\beta^a\beta^b, \tag{11}$$

This formulation of KK is non-standard from the textbook version, e.g., see [43], because it reverses the covariant and contravariant forms from the standard and inverts γ'_{44}. In addition, we do not equate β^a with the electromagnetic vector potential since it is needed for the gravitational theory. Thus, the formulation is equivalent to the ADM formalism [36] but in one additional dimension, and I will use ADM vocabulary such as referring to the β^a as the "shift vector" and α as the "lapse function" below.

Let $\alpha = e^{-3\phi}$. Define the Pauli metric, $\tilde{\gamma}_{AB}$, in terms of the Jordan metric: $\gamma'_{AB} = \alpha^{2/3}\tilde{\gamma}_{AB} = e^{-2\phi}\tilde{\gamma}_{AB}$.

$$\tilde{\gamma}_{44} = e^{5\phi} + \tilde{g}_{ab}\beta^a\beta^b, \tag{12}$$

$$\tilde{\gamma}_{4a} = e^{2\phi}\beta_a, \quad \tilde{\gamma}_{ab} = \tilde{g}_{ab}. \tag{13}$$

The KK action, under the conformal relationship between $\tilde{\gamma}$ and γ' is,

$$\begin{aligned} \mathcal{S}_{KK} &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\gamma'}{}^{(5)}R \\ &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\tilde{g}}\left({}^{(5)}\tilde{R} - \frac{16}{3}e^{\frac{3\phi}{2}}\Delta e^{-\frac{3\phi}{2}}\right), \end{aligned} \tag{14}$$

where Δ is the Laplace-Beltrami operator in 5-D.

This eliminates the dependence on α in the volume element from the action.

The Jordan metric's submanifold components have the form,

$$\gamma'^{ab} = e^{2\phi}\tilde{g}^{ab} + e^{-4\phi}\beta^a\beta^b. \tag{15}$$

We consider this to be the "physical" metric while the Pauli is the gravitational. A test mass with velocity vector u_A such that,

$$\gamma'^{AB}u_A u_B \tag{16}$$

defines its geodesic. In general it is possible that $u_4 \neq 4$ since the momenta in τ would contribute to the Kaluza-Klein particle mass. We neglect u_4 here, however. In this case, the geodesic is defined by Bekenstein-Sanders (1).

The KK action in the cylinder condition, which is enforced by the compactification, where all derivatives with respect to x_4 are zero, obeys the following relationship,

$$\begin{aligned} S_{KK} &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\gamma'^{(5)}}R' \\ &= \frac{1}{16\pi G}\int d^4x\,\frac{\sqrt{-g'}}{\alpha}\left({}^{(4)}R' + \frac{\alpha^2}{4}F'_{ab}F'^{ab}\right), \end{aligned} \tag{17}$$

where $F_{ab} = \nabla_a\beta_b - \nabla_b\beta_a$. The volume element is always real and non-negative.

Making the conformal transformation (14) to eliminate the lapse function from the volume element, the action is,

$$\begin{aligned} S_{KK} = \quad & \frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}} \\ & \left(K^{cdab}\nabla_c\beta_d\nabla_a\beta_b + {}^{(4)}R - \frac{16}{3}e^{\frac{3\phi}{2}}\Delta e^{-\frac{3\phi}{2}}\right) \end{aligned} \tag{18}$$

where $K^{cdab} = e^{-6\phi}(\frac{1}{2}\tilde{g}^{ac}\tilde{g}^{bd} + \frac{1}{2}\tilde{g}^{bc}\tilde{g}^{ad} - \tilde{g}^{ab}\tilde{g}^{cd})$.

Using $\Delta = \nabla^a\partial_a$ where $\nabla^a v_a = g^{ab}[\partial_b v_a + \Gamma^c{}_{ab}v_c]$,

$$e^{3\phi/2}\Delta e^{-3\phi/2} = -\frac{3}{2}\nabla^a\partial_a\phi + \frac{9}{4}\partial^a\phi\partial_a\phi.$$

The total covariance term, first term on the RHS, does not contribute to the equations of motion.

The shift functions are timelike $\beta^a\beta_a = -\beta^2$ where

$$\beta^2 = e^{6\phi} - e^{2\phi} \tag{19}$$

is a scalar function. This ensures that the universally coupled metric 15 has the same form as the Bekenstein-Sanders (1) since one may replace the vector with $A^a = \beta^a/\beta$ and pull out the factor of β^2.

The constraint has auxiliary field λ_β, and the action is now,

$$S_{KK} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \Big(K^{cdab} \nabla_c \beta_d \nabla_a \beta_b + {}^{(4)}R - 12 \partial^a \phi \partial_a \phi + \lambda_\beta (\beta^a \beta_a + \beta^2) \Big). \quad (20)$$

We can also write the term $K^{cdab} \nabla_c \beta_d \nabla_a \beta_b = e^{-6\phi} \frac{1}{2} g^{ac} g^{bd} F_{ab} F_{cd}$.

Add an auxiliary field μ_1. This has a separate action,

$$S_\mu = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\mu_1 \hat{g}^{ab} \partial_a \phi \partial_b \phi + U(\mu_1) \right], \quad (21)$$

where μ_1 is a non-dynamical, dimensionless scalar field and

$$\hat{g}^{ab} = \tilde{g}^{ab} - c_4 \beta^a \beta^b, \quad (22)$$

where c_4 is the constant in 5. The potential $U(\mu)$ is an arbitrary function that depends on a scale ℓ_B.

The auxiliary fields λ_β and μ_1 are not physical but instead represents a constraint on the shift vector β^a and scalar field, ϕ (which is a constraint on the lapse function α.) These in turn are constraints in the ADM formalism. Their function is to orient and scale the 3 + 1-D slices in the fifth dimension. Such constraints are common in ADM-based numerical relativity such as Bona-Massó slicing [41] with a "Gamma-driver" condition on the shift vector creating the highly successful "moving-puncture" conditions in simulations of inspiraling black holes [44].

The connection between the conformal rescaling and the slicing condition is a consequence of degrees of freedom in the metric. While the actions are all in terms of the unscaled Jordon metric, $\tilde{\gamma}$, rather than the rescaled Pauli metric, γ [43], the Jordan metric is not completely free for a choice of conditions. In numerical simulations of relativity using the ADM formalism, e.g., York's conformal dynamics slicing [42] and thin sandwich model [45], the conformal scale of the submanifold cannot be chosen arbitrarily but must be solved for given a choice of slicing. The same is true in higher dimensions. Hence, only the conformally invariant aspect of the submanifolds have unconstrained degrees of freedom. This suggests that slicing and conformal freedom are not separate degrees and that the Pauli metric representation is needed.

We now want to take the variation of S_{KK} with respect to the three fields ϕ, β^a, and g_{ab}. Write the action as a sum of actions. Let $S_{KK} = S_R + S_\beta + S_\phi$. Then the full action with the constraints is one for each term, $S_{KK\mu\lambda} = S_R + S_\beta + S_\phi + S_\lambda + S_\mu$.

We assume that matter is given by some Lagrangian dependent on a variety of fields $S_M = \int d^4x\, \sqrt{\tilde{\gamma}}\mathcal{L}_M[\tilde{\gamma}^{ab}, \beta^a, \phi, \Psi, A^a]$. In this section, we are only concerned with the stress-energy tensor $\delta S_M = -\frac{1}{2}\int d^4x\sqrt{-g}T_{ab}\delta g^{ab}$ where $T_{4a} = T_{44} = 0$.

Variation of $S_{KK\mu\lambda} + S_M$ with respect to the metric $\tilde{g}^{ab}$ leads to the field equations,

$$\begin{aligned}\tilde{G}_{ab} \;=\; & -e^{-6\phi}(\tilde{g}^{cd}F_{ca}F_{db} - \frac{1}{4}\tilde{g}_{ab}F^{cd}F_{cd}) + \\ & (12+\mu_1)\partial_a\phi\partial_b\phi + 6\tilde{g}_{ab}(U' - \beta^a\beta^b\partial_a\phi\partial_b\phi) \\ & -2\beta^c\partial_c\phi\beta_{(a}\partial_{b)}\phi - \lambda_\beta\beta_a\beta_b + \\ & 8\pi G[T_{ab} + 2e^{-6\phi}\beta^c T_{c(a}\beta_{b)}] + \frac{1}{2}(\mu U' - U)\tilde{g}_{ab} \end{aligned} \tag{23}$$

and we have used the slicing condition,

$$\hat{g}^{ab}\partial_a\phi\partial_b\phi = -U' \tag{24}$$

and $U' = \partial U/\partial\mu_1$.

For variations with respect to ϕ and β^a, one can apply the Euler-Lagrange and rearranging,

$$\begin{aligned}&\nabla_a([(12+\mu_1)\tilde{g}^{ab}\partial_b\phi + \mu_1\beta^a\beta^b\partial_b\phi) = \\ &8\pi G\left(\tilde{g}^{ab} + e^{-6\phi}\beta^a\beta^b\right)T_{ab} + 6K^{cdab}\nabla_c\beta_d\nabla_a\beta_b\end{aligned} \tag{25}$$

The final equation for the vector field is,

$$\begin{aligned}\nabla_c(e^{-6\phi}F^c{}_a) = \quad & -2\lambda_\beta\beta_a - \mu_1\beta^c\partial_c\phi\partial_a\phi + \\ & (8\pi G)e^{-6\phi}\beta^c T_{ca},\end{aligned} \tag{26}$$

which, by contracting with β^a, solves for λ_β,

$$\lambda_\beta = \frac{1}{2}\Big[-\beta^a\nabla_c(e^{-6\phi}F^c{}_a) - \mu_1\beta^a\beta^c\partial_a\phi\partial_c\phi - 8\pi G e^{-6\phi}\beta^a\beta^c T_{ac}\Big](e^{6\phi} - e^{2\phi})^{-1} \tag{27}$$

Thus, we have a total of 15 field equations.

One can easily recover general equivalence to 3+1-D GR by rescaling $\lambda_\phi\phi' = \phi$ and $\lambda_\phi \to 0$, in which case both the scalar and vector parts of the action go to zero. This is also equivalent to letting $\alpha^2 \to 0$ in the original Kaluza-Klein theory.

A. Quasi-static limit

Now we will show how to recover both Newtonian and MOND behavior in the case of slow motion and weak potentials. In the quasi-static limit, one expands $\phi = \phi_0 + \varphi$ where ϕ_0 is constant and φ is independent of time and $|\varphi| \ll 1$. The Einstein-Hilbert metric becomes $g_{00} = e^{-2\phi_0}(1 - 2\Psi)$ and $g_{ij} = e^{2\phi_0}(1-2\Theta)\gamma_{ij}$ where $\gamma_{ij} = \delta_{ij} + h_{ij}$. The shift vectors are $\beta_0 = -e^{-\phi_0}(e^{6\phi_0} - e^{2\phi_0})(1+\Psi)$, $\beta^0 = -e^{\phi_0}(e^{6\phi_0} - e^{2\phi_0})(-1+\Psi)$ and $\beta^i = \beta_i = 0$. The matter metric is then $\tilde{g}_{00} = -(1-2\tilde{\Psi})$ and $\tilde{g}_{ij} = (1-2\tilde{\Phi})\gamma_{ij}$ where $\tilde{\Psi} = \Psi - \varphi$ and $\tilde{\Phi} = \Phi - \varphi$.

The quasi-static limit has the same assumptions as the first Parameterized Post-Newtonian (1PPN) limit. The gravitational field is a small fluctuation about the background Minkowski space-time. Matter is represented with an effective perfect fluid with density ρ, pressure p, internal energy Π and 3-velocity $\vec{v}$. All fields are expanded perturbatively in orders of $v = |\vec{v}|$. Let Φ_ρ be the Poisson potential from the baryonic only matter density $\vec{\nabla}^2\Phi_\rho = 4\pi G_N \rho$ and G_N Newton's constant. As in the PPN formalism, $\partial/\partial t \sim O(v)$, $\Phi_\rho \sim \rho \sim \Pi \sim \varphi \sim O(v^2)$, $p \sim O(v^4)$. We also have $h_{ij} \sim O(v^2)$ and $\beta_i \sim O(v^3)$.

The quasi-static limit only contains terms up to $O(v^2)$ so terms containing p and $\rho\Pi$ are ignored. This means that matter is simply dust $T_{ab} = \rho u_a u_b$ with a normalized four velocity u^a.

With these assumptions, the spatial part of the equations 23 reduces to $G_{ij} = 0$ as in AeST. The disformal transformation gives $h_{ij} = 2[\varphi - \gamma_{PPN}\Phi_N]\delta_{ij}$ and shows that $\gamma_{PPN} = 1$. All the other terms are $O(v^3)$ or greater except those corresponding to the MOND equations [30][22]. Let $\mu = \mu_1 + 12$. Then,

$$\nabla^2\tilde{\Psi} = \frac{8\pi G}{2 - c_1 + c_4}\rho, \tag{28}$$

$$\nabla_i(\mu\nabla^i\varphi) = 8\pi G\rho, \tag{29}$$

$$\tilde{\Phi} = \tilde{\Psi} \tag{30}$$

where in this case $c_1 = \frac{1}{2}(1 - e^{-4\phi_0})$ and c_4 is chosen in the constraint 22.

Define $\partial_\perp = (\partial_4 - \mathcal{L}_\beta)$, where $\mathcal{L}_\beta$ is the Lie derivative along the shift vector, as a derivative perpendicular to the 4D submanifold.

For a scalar field, $\mathcal{L}_\beta = \beta^a\partial_a\phi$, and, for example, Bona-Massó slicing imposes a condition on the the lapse function, $\partial_\perp\alpha = -\alpha^2 a(\alpha)K$ where K is the trace of the extrinsic curvature and a is a function that gives the desired slicing. Slicing conditions, in general, are expressed as an equation involving a derivative of how the lapse function changes perpendicular to the submanifold,

$\partial_\perp \alpha = f(\alpha, \partial_a \alpha)$.

Given that we assume $\partial_4 \cdot = 0$, and our MOND slicing condition (24) can be rewritten,

$$\partial_\perp \phi = \pm\sqrt{\frac{dU}{d\mu} + \partial^a \phi \partial_a \phi} \tag{31}$$

Since $\phi = -\frac{1}{3}\log|\alpha|$ and

$$\partial_a \phi = \frac{1}{3\alpha}\partial_a \alpha$$

we can make a substitution so that this becomes a condition on the lapse function directly. If we let

$$U = \left(\frac{1}{3\alpha}\right)^2 U_\alpha \tag{32}$$

then the condition on the lapse function is

$$\partial_\perp \alpha = \pm\sqrt{\frac{dU_\alpha}{d\mu} + \partial^a \alpha \partial_a \alpha}. \tag{33}$$

This is now written as a slicing condition perpendicular to the submanifold.

Equation 33, combined with the condition on β^a, 19, now fixes the 5D theory.

To quasi-static order, $U \sim V$, where V is the standard AeST potential given in 3. MOND defines μ as $\mu = \frac{df}{dX}$ where $X = \ell_B^2 \hat{g}^{ab}\partial_a \phi \partial_b \phi$ and $f = \mu X + \ell_B^2 V$. It does not precisely specify μ or V; it only determines two limits. MOND is achieved if,

$$\frac{dV}{d\mu} \to -\frac{4}{9b_0^2 \ell_B^2}\mu^2$$

where b_0 is a constant determined by ϕ_0 and the MOND acceleration parameter a_0 [30]. Meanwhile, it diverges for $\mu \to \mu_0$, for example, $\frac{dV}{d\mu} \to (\mu_0 - \mu)^{-m}$, gives the Newtonian limit where μ_0 is a constant.

Since μ is a function of X, this means that perpendicular to the submanifold in the Newtonian limit,

$$\partial_\perp \phi \approx \pm\sqrt{\frac{dV}{d\mu}} \to \infty.$$

This is true if $\alpha \to 0$ according to 32 which is consistent with standard General Relativity (the vector potential drops out of the equations). This occurs if the scalar field is very large, $\phi \to \infty$.

In the MOND limit, on the other hand, the slicing condition is

$$\partial_\perp \phi \to \pm\sqrt{\partial^a \phi \partial_a \phi - \frac{4}{9b_0^2 \ell_B^2}\mu^2} \tag{34}$$

where under spherical symmetry, $\mu \to \frac{2G_N}{G}\frac{1}{\ell_B a_0}e^{\phi_0}\sqrt{\ell_B^2 \hat{g}^{ab}\partial_a\phi\partial_b\phi}$ and G_N depends on μ_0 and ϕ_0 [30].

Once the degrees of freedom are selected, the scalar field's spacetime distribution is determined by 25 with the primary influence being the baryonic matter distribution. Its evolution in the 5th dimension is determined by the slicing condition 31.

The slicing condition is the source of MOND phenomenon. The equation 31 is in terms only of scalar quantities with respect to the 3+1-D manifold. Thus, it is invariant under 3+1-D coordinate transformations.

IV. FLRW BACKGROUNDS AND SPIN-2 TENSOR PROPAGATION

AeST requires fine tuning (7) that ensures that it matches empirical data against Friedmann-Lemaître-Robertson-Walker (FLRW) backgrounds. In particular, the tensor mode propagation speed and Shapiro delay must be the same as electromagnetic waves to agree with observations of GW170817 [37]. This fine tuning is *ad hoc* in AeST but predicted in the KK theory.

The prediction comes from the unique determination of the Bekenstein-Sanders metric (1). Sanders showed that the metric's form (up to a constant scaling of the scalar field) is determined by global conformal symmetry and the independence of units of the fine structure constant [46]. Indeed, the form is uniquely determined. By enforcing the form of the metric to agree with this, however, the KK theory automatically predicts the fine tuning used in [22].

The key observation is in the differences between the equations for AeST (as derived from 3 and given in [22]) and those of the KK theory's equation for the Einstein tensor (23).

Let the universally coupled metric g_{ab} have the following form,

$$ds^2 = -dt^2 + a^2(\gamma_{ij} + \chi_{ij})dx^i dx^j. \tag{35}$$

In this case, a is the scale factor, γ_{ij} is the spatial metric of constant curvature κ, and χ_{ij} is the transverse-traceless tensor mode GW such that $\gamma^{ij}\chi_{ij} = 0$ and $\nabla_i \chi^i{}_j = 0$.

In the case of tensor modes, ϕ and the shift β^a are unperturbed, $\beta^i = 0$, and $\phi = \bar{\phi}(t)$, the spatial derivatives in ϕ drop out.

To show that the Kaluza-Klein equations for tensor mode propagation are the same as AeST. We could go through the lengthy and tedious computations which can be found in [47], or we exploit a property of FLRW spacetime to show equivalence trivially. The later approach gets around the fact that the KK shift vector has an additional factor dependent on the lapse function not present in AeST. The key is to remove any derivatives of β^a from the action (20) for FLRW backgrounds in which case the factor can be pulled out as shown in the following:

By construction, we have β^a timelike orthogonal to the 3D spatial hypersurface. This is a non-trivial affinely parameterized geodesic vector field. By the Frobenius theorem, the twist tensor is naught: $F_{ab} = \nabla_{[a}\beta_{b]} = 0$. Therefore, terms dependent on F_{ab} drop out of the action (20). Hence, the action is only left with terms that have no derivatives of the shift vector β^a. Now, replace the time oriented component β^0 with $(e^{6\bar{\phi}} - e^{2\bar{\phi}})A^0$ where A^0 is the time component of the AeST vector field. The factor of $(e^{6\bar{\phi}} - e^{2\bar{\phi}})$ multiplies with the factors of $e^{-6\bar{\phi}}$ from the lapse function in the action to give a factor $1 - e^{-4\bar{\phi}}$ which matches the factor in the AeST action [22]. Thus, in the perturbation theory of spin-2 tensor modes against an FLRW background, the two theories have the same action.

Let $A_0 = -e^{-\bar{\phi}}$. This means that the tensor mode equations are the same in both theories in the perturbation theory. The tensor mode obeys [22],

$$e^{2\bar{\phi}}(1 - c_{13})[\ddot{\chi}^i{}_j + (3H + 4\dot{\bar{\phi}})\dot{\chi}^i{}_j] - e^{2\bar{\phi}}\frac{dc_{13}}{d\phi}\dot{\bar{\phi}}\dot{\chi}^i{}_j - \frac{1}{a^2}e^{-2\bar{\phi}}(\nabla^2 - 2\kappa)\chi^i{}_j = 16\pi G e^{-2\bar{\phi}}\Sigma^{(g)i}{}_j. \tag{36}$$

Here $H = \dot{b}/b$ is the rescaled Hubble parameter where $b = ae^{\phi}$ and $\Sigma^{(g)i}{}_j$ is a traceless matter term related to anistropic stress [47]. From 6, $c_{13} = 1 - e^{-4\phi}$ ensures the correct fine tuning. For the KK theory, the product of the square lapse function and the square magnitude of the shift gives this value $-\alpha^2\beta^2 = c_{13}$. This is precisely the square of the magnitude of a shift of a point x^a for a temporal distance $\alpha d\tau$ in the fifth dimension required by the Bekenstein-Sanders form of the metric.

This means that, while in AeST the value of c_{13} is chosen empirically as a free function, in the KK theory c_{13} is not free at all but fixed by the Einstein-Hilbert action. Nevertheless, they are the same in both theories.

V. THE IMPLICATIONS OF EXTRA DIMENSIONS

The lightest Kaluza-Klein particle remains a strong candidate for WIMPs [48]. The Kaluza-Klein dark matter theory proposes that CDM arises from the momenta propagating in a higher dimension [39]. Such momenta produces a ladder of particles, n/R, where R is the compactified dimension radius and n is the mode. The lightest on order $1/R$ is stable and a good candidate for a Weakly Interacting Massive Particle (WIMP). LKPs arise from a minimal model of Universal Extra Dimensions (MUED) where the LKP is a massive hypercharge gauge boson. They are expected to have a mass of 500-2000 GeV depending on the effective annihilation cross section. In the MUED KK particle theory, the scalar of the higher dimensional graviton tensor, which I will show is the MOND potential, is referred to as the radion field [49]. The massive $n = 1$ radion particle is generally ignored in MUED. So far no evidence of LKP has been found. Since the scalar and vector fields, meanwhile, arise from the $n = 0$ modes in a compactified dimension, there are no particles necessarily to find.

Direct detection experiments so far have been insensitive to the heavier, TeV LKP. It is unlikely that this will change in the near future. Indirect detection through gamma rays, neutrinos and synchrotron flux, positrons, antiprotons, and antideuterons have also been looked at [50]. At the Large Hadron Collider (LHC) the main method of detecting LKP would be from strongly produced KK gluons and KK quarks.

That KK theory implies the existence of a vector and scalar field as well as modes that generate massive particles is uncontroversial. What has generally been unknown is the properties of those fields and particles. In this paper, it has been asserted that the vector and scalar fields appear as additional gravitational fields that act at large scales but not at small. The reason they are not measurable at the Solar System scale is because of the nature of the slicing condition. The closer to constant slicing the more gravity will be Newtonian. It is the variations in slicing from location to location that generates the MOND potential and lensing effects from the shift vector. At Solar System scales, the accelerations are high enough that the slicing becomes nearly constant.

The extra dimension need not be compactified. Another compelling explanation is that $\partial_4 \cdot \approx 0$ for classical phenomenon but at the quantum level the universe has stochastic [51] or chaotic [52] flow in the 5th dimension, i.e., the fifth dimension is non-compactified [43].

Rovelli has defined a covariant mechanism for an equilibrium flow dimension in terms of a "multifingered" thermal time parameter as the "speed" of time [53] based on a symplectic manifold [54]. In the stochastic quantization description, under Wick rotation for a Gibbs state $\rho_\beta = Z^{-1}(\beta)e^{-\beta H}$, for $Z(\beta) = \int e^{-\beta H}$ and H the ADM Hamiltonian for the 4D geometry as it flows in τ, this thermal parameter is uniformly $1/\beta = \hbar$, which is the ratio of thermal parameter $\mathfrak{t}$ to physical parameter τ, $\hbar = \mathfrak{t}/\tau$. Based on the spectral lines of distant galaxies, this ratio must be constant to within very tight constraints, but the physical τ parameter can vary according to the scalar field as long as the thermal parameter varies by the same amount.

In the stochastic interpretation, neglecting matter, if we allow the 4D submanifold be a de Sitter space, then under Wick rotation it is the surface of a 4-sphere, S^4 with radius $\sqrt{3/\Lambda}$ where Λ is the cosmological constant. This closed submanifold is one potential slicing of an FLRW spacetime in one higher dimension.

If the overall manifold is anti-deSitter, i.e., AdS_5 symmetry, having a negative cosmological constant, it connects to $N = 4$ super Yang-Mills theory [55]. Such a connection would resolve two problems with the AdS/CFT correspondence: that it is not realistic because our universe is four dimensional and that our universe appears to be de Sitter. An investigation of the possibility of a negative cosmological constant and Anti-deSitter symmetry is left for future work.

Since slicing is simply a choice of coordinates, it comes down to an observer's state of motion relative to the extra dimension. This is given by 2. Choose a point in spacetime x^a. A distant observer, O, measures a field as it propagates from τ to $\tau + d\tau$. The observer sees that its location in spacetime, x^a shifts by the shift vector $x^a + \beta^a d\tau$. In addition, compared to proper time, time propagates at a rate of α. For every infinitesimal clock tick in proper time, the field propagates at $x^a + \alpha\beta^a d\tau$. Thus, the slicing is a property of how fields propagate within the compactified dimension and that in turn depends on a combination of the distribution of matter and the interpolating function, μ.

Since the slicing condition is analogous to a gauge fixing, it means that parameters such as a_0 and the form of μ may not be fixed as well. We do not explore that possibility in this paper but that is an avenue for further investigation.

The vector field is analogous to the gravitomagnetic potential in GEM. GEM is well understood when applied to 4-D GR and the same principles can be applied in 5D [56]. The gravitoelectric field from the lapse function, $G^a = -\nabla^a \phi$, for example produces MOND effects while the Newtonian force comes from the time-time component of the metric. Meanwhile, the lensing is due to non-linear variations in the shift vector. The Sagnac effect is an example of a well understood effect on light that comes directly from a shift vector in 4D GR. Tidal forces, differential dragging, described by the gravitomagnetic tidal tensor $\mathbb{H}_{AB} \sim \partial_a \partial_b \beta_c$ may also occur.

Ordinary post-Newtonian GEM cannot explain galactic rotation curves [57][58], despite some flawed attempts to do so [59], because their contributions are order $(v/c)^2$. But higher dimensional GEM can because it can generate much stronger gravitational effects on the same order as the Newtonian force.

This suggests possible tests for the theory involving higher dimensional gravitomagnetic effects similar to the Lense-Thirring and tidal effects that dark matter would not cause.

VI. CONCLUSION

In conclusion, I have shown that a Kaluza-Klein theory in 5D can replicate Aether-Scalar-Tensor theory without ruling out additional sources of dark matter from MUED Kaluza-Klein dark matter WIMPs. I had also shown that the KK theory constrains most of the freedom in the definition of AeST. The only freedom is in the slicing condition which is responsible for MOND and must be matched to empirical data.

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